

**COMMENT:****On the thermodynamics of inhomogeneous perfect fluid mixtures****Bartolomé Coll<sup>1</sup> and Joan Josep Ferrando<sup>2</sup>**<sup>1</sup> Systèmes de référence relativistes, SYRTE, Observatoire de Paris-CNRS, 75005 Paris, France.<sup>2</sup> Departament d'Astronomia i Astrofísica, Universitat de València, 46100 Burjassot, València, Spain.E-mail: [bartolome.coll@obspm.fr](mailto:bartolome.coll@obspm.fr); [joan.ferrando@uv.es](mailto:joan.ferrando@uv.es)

**Abstract.** It is shown that inhomogeneous Szekeres and Stephani universes exist corresponding to non-dissipative binary mixtures of perfect fluids in local thermal equilibrium. This result contradicts a recent statement by Zárate and Quevedo (2004 *Class. Quantum Grav.* **21** 197, *Preprint* gr-qc/0310087), which affirms that the only Szekeres and Stephani universes compatible with these fluids are the homogeneous Friedmann-Robertson-Walker models. Thus, contrarily to their conclusion, their thermodynamic scheme do not give new indications of incompatibility between thermodynamics and relativity. Two of the points that have generated this error are commented.

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**1. Thermodynamic scheme for a mixture of two perfect fluids**

In a recent paper, Zárate and Quevedo [1] extend the standard thermodynamic scheme, of local thermal equilibrium for a simple perfect fluid, to the case of a mixture of two perfect fluids.

They define their thermodynamic scheme by the following relations:

(a) conservation of the perfect fluid energy-momentum tensor:

$$\nabla \cdot T = 0, \quad T = (\rho + p)u \otimes u + pg, \quad (1)$$

where  $\rho$  is the total energy density,  $p$  the mixture pressure and  $u$  the 4-velocity of the total matter flow.

(b) conservation of the total current density:

$$\nabla \cdot (nu) = 0, \quad (2)$$

where  $n$  is the total particle number density.

(c) Gibbs thermodynamic relation:

$$T d s = d(\rho/n) + p d(1/n) - \mu d c, \quad (3)$$

where  $s$  is the entropy per particle,  $T$  the temperature,  $c$  the fractional concentration of one of the two perfect fluid components and  $\mu$  is the mixture chemical potential, i.e. the difference between the chemical potentials of the components of fractional concentration  $c$  and  $1 - c$  respectively.

Under these relations, and as Zárate and Quevedo emphasize, the entropy production no longer vanishes but is a consequence of the change in the fractional concentrations of the components.

In addition, this thermodynamic scheme for a mixture of two perfect fluids recovers the standard scheme for a simple fluid when  $c = \text{constant}$ . In this sense their scheme is *more general* than the standard one.

Nevertheless, its physical applicability risks to be much *more restrictive* than that of a simple fluid: the physical applicability of the non-dissipative (relativistic) evolution hypothesis clearly diminishes generically when in addition  $c \neq \text{constant}$ , i.e. the fluid is submitted to endothermic or exothermic reactions. A more realistic model would be consisted in a mixture of two perfect fluids generating a semi-perfect fluid, that is to say, a dissipative Pascalian one, with eventually a heat flux proportional to the gradient of the fractional concentration.

Anyway, as no complete set of constraints is known on the space of formal constitutive relations in continuous thermodynamics (i.e. the more or less general inequalities on the thermodynamic variables and on some of their partial derivatives are insufficient to separate physical from unphysical equations of state), it makes sense, as Zárate and Quevedo emphasize, to analyze the compatibility of their thermodynamic scheme with Einstein equations.

## 2. Compatibility of Zárate and Quevedo's thermodynamic scheme with Einstein equations

In the article [1] that motivates this comment, the authors quote a work by us [2] and they assert: "Coll and Ferrando have shown that an exact solution [to Einstein equations] admits a thermodynamic scheme provided the integrability conditions of Gibbs equation are satisfied".

It is here pertinent to observe that, of course, this fact is the starting point of the quoted paper, but that our main goal there is *i*) to obtain these integrability conditions explicitly and in terms of the sole hydrodynamic variables  $(u, \rho, p)$ , and *ii*) to interpret them physically. Our corresponding result is [2] (see also [3]): *i) a divergence-free perfect fluid energy tensor evolves in local thermal equilibrium if, and only if, the space-time function  $\xi \equiv \dot{\rho}/\dot{p}$  depends only on the variables  $\rho$  and  $p$ :  $d\xi \wedge d\rho \wedge dp = 0$ ; ii) then  $\xi$  is a state variable,  $\xi = \xi(\rho, p)$ , and represents the square of the velocity of the sound.*

As, on one hand, Einstein equations are biunivocally related to the sole hydrodynamic variables  $(u, \rho, p)$  and, on the other hand, the condition  $d\xi \wedge d\rho \wedge dp = 0$  is not a consequence of them, it follows that: *there exist perfect fluid solutions to Einstein equations that do not admit a standard thermodynamic scheme, i.e. that cannot be interpreted as evolutions, in local thermal equilibrium, of a single fluid.*

What is the corresponding result for Zárate and Quevedo's thermodynamic scheme for a non-dissipative mixture of two fluids?

As there exists only one scalar constraint for the compatibility between Einstein equations and the relatively restrictive standard thermodynamic scheme (namely, the one expressing that the quotient  $\dot{\rho}/\dot{p}$  is a function of state:  $\dot{\rho}/\dot{p} = \xi(\rho, p)$ ), it seems that the corresponding result for Zárate and Quevedo's scheme would be the absence of constraint, because of its less restrictive character. But this appears as a startling evaluation facing Zárate and Quevedo's main statement in [1]. Let us see it in detail.

Suppose given, in a domain of the space-time, a solution to Einstein equations for a perfect fluid,  $(u, \rho, p)$  (in fact, a solution to the divergence-free equations (1), the argument that follows being also valid for test fluids). Then,  $u$ ,  $\rho$  and  $p$  are known quantities in that domain and, in particular, the conservation equation for the total current density (2) becomes a linear and homogeneous scalar equation in the scalar unknown  $n$ . Pick in it a particular solution  $n$ ; with these four known elements  $(u, \rho, p, n)$  we can evaluate the one-form  $\omega$  defined by

$$\omega \equiv d(\rho/n) + p d(1/n) \quad (4)$$

Then, according to Gibbs equation (3) for the mixture, one has to explore the existence of the four thermodynamic scalars  $(s, c, T, \mu)$  such that

$$\omega = T ds + \mu dc \quad (5)$$

But this existence is exactly what Pfaff decomposition theorem locally guarantees in four dimensions for *any* one-form  $\omega$ , and *a fortiori* for the restricted one given by (4); we have thus:

**Proposition 1** *Any perfect fluid solution to Einstein equations is compatible with a Zárate and Quevedo's binary thermodynamic scheme.*

How many Zárate and Quevedo's schemes may be associate to any given perfect fluid solution to Einstein equations?

From the data  $(u, \rho, p)$  obtained as a solution to equation (1), the solution to the density conservation (2) is obviously determined up to an  $u$ -invariant function  $f$ ,  $\dot{f} \equiv u^\alpha \partial_\alpha f = 0$ , so that  $n = fn_o$ ,  $n_o$  being a particular solution. And for every such  $n$ , equation (4) determines the particular one-form  $\omega$ . Then, equation (5) defines generically the four thermodynamic variables  $s$ ,  $c$ ,  $T$ ,  $\mu$ , as functions of the three ones  $\rho$ ,  $p$ ,  $n$ ; the functions  $s$  and  $c$  result thus involved by one first order differential equation, namely

$$c'_p s'_n - c'_n s'_p = -\frac{\rho + p}{n} (c'_p s'_\rho - c'_\rho s'_p) , \quad (6)$$

which always admits solutions in one of the unknowns for every arbitrary choice of the other, such solutions depending (for example *via* an initial problem) on an arbitrary function of two of the three variables. Then the variables  $T$  and  $\mu$  are univocally given by

$$T = \frac{c'_p}{h} , \quad \mu = -\frac{s'_p}{h} \quad (7)$$

where

$$h \equiv n(s'_\rho c'_p - s'_p c'_\rho) . \quad (8)$$

**Proposition 2** *The different Zárate and Quevedo's binary thermodynamic schemes that any given perfect fluid solution  $(u, \rho, p)$  to Einstein equations admits, are obtained by the free choice of a  $u$ -invariant function (determination of the total particle number density  $n$ ), of a function of three variables (say, the entropy per particle,  $s(\rho, p, n)$ ) and of a function of two variables (say, the fractional concentration at a given value  $k$  of  $n$ ,  $c(\rho, p, k)$ ).*

The main statement in [1] asserts that *among the Szekeres and Stephani families of perfect fluid cosmological models, the only ones that admit a [Zárate and Quevedo's] binary thermodynamic scheme are the Friedmann-Robertson-Walker models.* Proposition 1 shows that this statement is wrong. Proposition 2 gives an indication of the 'distance' between this statement and the correct one.

As already said, no complete constraints are known on the space of formal constitutive relations insuring the physical character of a model; nevertheless, the richness of the choice of Zárate and Quevedo's schemes stated in Proposition 2 locally guarantees the usual thermodynamic inequalities (such as  $T > 0$  or  $0 \leq c \leq 1$ ).

### 3. Some Zárate and Quevedo's thermodynamic schemes for Szekeres and Stephani universes

The Szekeres and Stephani universes admitting a standard thermodynamic scheme, i.e. that of a one-component perfect fluid, have been considered by different authors. Bona and Coll [4] have shown that the Stephani universes admitting a thermodynamic scheme are space-times with a 3-dimensional group of isometries acting on 2-dimensional orbits. This result has been recovered in [5] where the authors have also studied the thermodynamic Szekeres-Szafron models, and they have shown that a family of thermodynamic solutions of class II without symmetries exists. A different approach has been used in [3] in order to study the Szekeres-Szafron space-times of class II: the solutions which represent a perfect fluid in local thermal equilibrium have been obtained and the associated thermodynamic schemes explicitly given. More specific thermodynamic analysis have been considered in other works. Thus, Sussman [6] has presented a family of Stephani universes which admit, up to a good approximation, an interpretation as classical mono-atomic ideal gases or as matter-radiation mixtures. On

the other hand, we have obtained all the Stephani universes which represent a generic ideal gas in local thermal equilibrium [7].

It is worth pointing out that, although in some cases the standard thermodynamic scheme imposes symmetries on the metrics, these papers show that *inhomogeneous Szekeres and Stephani universes exist and are known, that admit a standard thermodynamic scheme*.

On the other hand, Propositions 1 and 2 show that *all* perfect fluid space-times admit Zárate and Quevedo's thermodynamic schemes, but they give no explicit solutions to them. The difficulties to find such explicit solutions associated with a general perfect fluid lie only in the solution of the two differential equations (2) and (6) because, as we have seen, then the entropy per particle  $s$  (or equivalently the fractional concentration  $c$ ) may be chosen arbitrarily, and the temperature  $T$  and the mixture chemical potential  $\mu$  are explicitly given by (7) and (8).

Nevertheless, it is not difficult to find explicit Zárate and Quevedo's thermodynamic schemes for any perfect fluid admitting a standard thermodynamic one. Remember that a standard thermodynamic scheme is a perfect fluid solution  $(u, \rho, p)$  to equation (1) that admits a solution  $n$  to equation (2) and for which there exist functions  $\bar{s}$  and  $T$  of, say, the variables  $\rho$  and  $n$ , verifying

$$T d\bar{s} = d(\rho/n) + p d(1/n) . \quad (9)$$

Let  $(u, \rho, p, n, \bar{s}, T)$  be such a standard thermodynamic scheme. Choose a space-time function  $c = c(x^\alpha)$  satisfying  $0 \leq c(x^\alpha) \leq 1$ , but otherwise arbitrary, let  $\Phi(c)$  be an arbitrary real function of  $c$ , and define a new function  $s$  of  $\rho$ ,  $n$  and  $c$  by

$$s = s(\rho, n, c) \equiv \bar{s}(\rho, n) + \Phi(c) , \quad (10)$$

Then, on account of (9) one has:

$$T d s = d(\rho/n) + p d(1/n) + T \Phi'(c) d c \quad (11)$$

and, consequently, calling

$$\mu = -T \Phi'(c) , \quad (12)$$

the set of variables  $(u, \rho, p, n, s, T, c, \mu)$  defines a Zárate and Quevedo's thermodynamic scheme.

Consider then the Szekeres and Stephani universes that admit a standard thermodynamic scheme and that are explicitly given in the above mentioned references [3-7]. On every one of them, every choice of a pair of functions  $c(x^\alpha)$  and  $\Phi(c)$  directly defines, by means of equations (10) and (12), a Zárate and Quevedo's thermodynamic scheme, i.e. an explicit counterexample of the main statement of [1].

#### 4. Remarks

Perhaps it is worthwhile to indicate a pair of incorrect arguments in Zárate and Quevedo's work [1] that could explain their erroneous conclusion.

In dealing with Szekeres universes, they select, by integration of a space-like equation, the sole solution  $\mu = (\rho + p)/n$ , neglecting an arbitrary function of time (see their equation (27) in [1]). They neglect it under the argument that  $\mu$  being a function of three thermodynamic variables, and the expression  $\mu = (\rho + p)/n$  already containing them, one cannot add to  $\mu$  an arbitrary function of time without increasing the number of independent variables. This argument is incorrect because it mix up considerations on space-time variables and thermodynamic ones without taking care of their specific connection: the pressure being a function of the sole time in Szekeres space-times, to add a function of time is nothing but to add a function of the pressure  $p$ , an addition which is perfectly admissible in their situation but that invalidates their particular expression of Gibbs equation (their equation (28) in [1]) and their consequences.

In dealing with Stephani universes, they impose to them two equations of state, for  $\mu$  and  $s$  (their equation (40) in [1]), obtained in the Szekeres case under *i*) the abusive restriction above mentioned and *ii*) the hypothesis of spatially homogeneous pressure,  $p_{,i} = 0$ , identically satisfied in Szekeres universes but generically inadmissible in Stephani ones.

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